Further results on the hyperbolic Voronoi diagrams

Frank Nielsen
Ecole Polytechnique, France
Sony Computer Science Laboratories, Japan
Email: Frank.Nielsen@acm.org

Richard Nock
NICTA, Australia
UAG CEREGMIA, France
Email: rnock@martinique.univ-ag.fr

Abstract—In Euclidean geometry, it is well-known that the k-order Voronoi diagram in \( \mathbb{R}^d \) can be computed from the vertical projection of the k-level of an arrangement of hyperplanes tangent to a convex potential function in \( \mathbb{R}^{d+1} \); the paraboloid. Similarly, we report for the Klein ball model of hyperbolic geometry such a concave potential function: the northern hemisphere. Furthermore, we also show how to build the hyperbolic k-order diagrams as equivalent clipped power diagrams in \( \mathbb{R}^d \). We investigate the hyperbolic Voronoi diagram in the hyperboloid model and show how it reduces to a Klein-type model using central projections.

Keywords—Voronoi diagram; hyperbolic geometry; clipping.

I. INTRODUCTION

Hyperbolic geometry is a consistent geometry where the Euclidean Playfair’s parallel postulate is discarded and replaced by the existence of many lines \( U \) not intersecting another given line \( L \) and passing through a given point \( P \notin L \) (the \( U \)'s are said ultra-parallel\(^1\) to \( L \)). Hyperbolic geometry can be studied using various models [1]: Poincaré disk or upper plane conformal models, Klein non-conformal model disk model, hyperboloid conformal model, etc. From the viewpoint of computational geometry, we prefer to use Klein model where lines/bisectors are Euclidean straight [2] and then convert the output to the desired model for visualization or navigation purposes [1]. We report further novel results for constructing hyperbolic Voronoi diagrams (HVDs) in Klein model [2] and present yet another approach to get Klein-type affine bisectors/diagrams from the hyperboloid\(^2\) model.

II. HVDs FROM LOWER ENVELOPES

The Voronoi diagram of a set \( \mathcal{P} = \{p_1, \ldots, p_n\} \) of \( n \) points in \( \mathbb{R}^d \) w.r.t. \( D(\cdot, \cdot) \) can be computed equivalently as the minimization diagram of \( n \) functions by observing that \( D(x, p_i) \leq D(x, p_j) \iff F_i(x) \leq F_j(x) \) where \( F_i(x) = D(x, p_i), i \in \{1, \ldots, n\} \). Thus the combinatorial structures are congruent: \( \text{Vor}_D(\mathcal{P}) \cong \min_{i \in \{1, \ldots, n\}} F_i(x) \). Furthermore, this minimization diagram amounts to compute the lower envelope of \( n \) graph functions in \( \mathbb{R}^{d+1} \):

\[
\mathcal{F}_l : \{(x, y = F_i(x)) : x \in \mathbb{R}^d\}.
\]

\(^1\)Parallel lines intersect at infinity in hyperbolic geometry.

\(^2\)Hyperbolic geometry stems from the hyperboloid model.

Let \( \langle x, p \rangle = x^T p = \sum_{i=1}^d x^{(i)} p^{(i)} \) denotes the Euclidean inner product. In the Klein model [2], the distance between two points \( x \) and \( p \) in the open unit ball domain \( \mathbb{B}_d = \{x \in \mathbb{R}^d : \langle x, x \rangle < 1\} \) is \( D^K(x, p) = \arccosh\left(\frac{1 - \langle x, p \rangle}{\sqrt{1 - \langle x, x \rangle} \sqrt{1 - \langle p, p \rangle}}\right) \) where \( \arccosh(x) = \log(x + \sqrt{x^2 - 1}) \) for \( x \geq 1 \) is a monotonically increasing function. Since the Voronoi diagram does not change by composing the distance with a monotonous function, we consider the equivalent Klein distance \( d^K(x, p) = \frac{1}{\sqrt{1 - \langle x, x \rangle} \sqrt{1 - \langle p, p \rangle}} \). To each point \( p_i \in \mathcal{P} \) corresponds a function \( F_i(x) = d^K(x, p_i) \). Since the denominator \( \sqrt{1 - \langle x, x \rangle} \) is common to all functions, the minimization diagram is equivalent to the minimization diagram of \( F_i^0(x) = \frac{1}{\sqrt{1 - \langle p, p \rangle}} \). The graph \( \mathcal{F}_l^0 = \{(x, y = F_i^0(x)) : x \in \mathbb{B}_d\} \) are hyperplanes in \( \mathbb{R}^{d+1} \) defined on \( \mathbb{B}_d \), and the lower envelope can thus be computed from the intersection of \( n \) half-spaces \( H_i^0 : y \leq \frac{1}{\sqrt{1 - \langle p, p \rangle}} \), yielding the Voronoi unbounded polytope in \( \mathbb{R}^{d+1} \).

Theorem 1: The HVD of \( n \) points can be computed in the Klein model as the intersection of \( n \) half-spaces in \( \mathbb{R}^{d+1} \) and by projecting vertically (\( \downarrow H_0 : y = 0 \)) the polytope on \( \mathbb{R}^d \), and clipping it with the unit ball domain: \( \text{Vor}_{d,K}(\mathcal{P}) = ((\cap_{i=1}^n H_i^0) \downarrow H_0) \cap \mathbb{B}_d \).

III. LIFTING SITES TO A POTENTIAL FUNCTION

In Euclidean (and more generally Bregman geometry), the Voronoi polytope is built by lifting points to tangent hyperplanes to a potential function \( y = F(x) \) at site locations. This is the paraboloid lifting transformation: \( y = F(x) = \langle x, x \rangle \) \( (y = F(x) \) for a convex Bregman generator \( F) \).

Theorem 2: In the Klein ball model, the potential function for lifting generators to hyperplanes is the concave function \( y = F(x) = \sqrt{1 - \langle x, x \rangle} \) restricted to \( \mathbb{B}_d \).

Proof: Let us identify the hyperplane equation \( H(p) : y = \frac{1}{\sqrt{1 - \langle p, p \rangle}} \) with the hyperplane tangent at \( p \) to a potential function \( y = F(x) : \langle \nabla F(p), x - p \rangle + F(p) = \langle x, \nabla F(p) \rangle + F(p) - \langle p, \nabla F(p) \rangle \). We have \( \nabla F(p) = \frac{p}{\sqrt{1 - \langle p, p \rangle}} \) and the remaining term (independent of \( x \)) is \( \frac{1}{\sqrt{1 - \langle p, p \rangle}} \). The anti-derivative of \( \nabla F(x) = -\frac{x}{\sqrt{1 - \langle x, x \rangle}} \) is \( \sqrt{1 - \langle x, x \rangle} + c \), and
the constant $c$ solves to zero. This is the equation $y^2 + \langle x, x \rangle = 1$ of the northern hemisphere for $y \geq 0$. Observe that the hyperplanes tend to become vertical as we near the boundary domain $\partial \mathbb{B}_d$, and are vertical at the boundary.

IV. $k$-ORDER HYPERBOLIC VORONOI DIAGRAMS

Since the Klein bisector is affine, the $k$-order HVD is affine. We present two construction methods.

A. $k$-HVDs from layers of an arrangement of hyperplanes

This is a straightforward generalization of the Euclidean procedure using the $\sqrt{1-\langle x, x \rangle}$ potential function. The $k$-order HVD is a cell complex that can be built by projecting to $\mathbb{R}^d$ all the $(d+1)$-dimensional cells at $k$-level of the arrangement of the site hyperplanes $\mathcal{H} = \{H_1, \ldots, H_n\}$ of $\mathbb{R}^{d+1}$ and clipping the structure to $\partial \mathbb{B}_d$. Figure 1 displays some $k$-order diagrams and illustrates some degenerate cases.

B. $k$-HVDs from power diagrams

Consider all subsets of size $k$, $\mathcal{P}_k = \binom{\mathcal{K}}{k} = \{\mathcal{K}_1, \ldots, \mathcal{K}_N\}$ with $N = \binom{n}{k}$. Those subset generators partition the space into non-empty $k$-order Voronoi cells:

$$\text{Vor}_k(\mathcal{K}) = \{x : \forall q \in \mathcal{K}_i, \forall r \in \mathcal{P}\setminus\mathcal{K}_i, D(x, q) \leq D(x, r)\}.$$  

Observe that $x \in \text{Vor}_k(\mathcal{K})$ iff $\sum_{p \in \mathcal{K}_i} D(x, p) \leq \sum_{p' \in \mathcal{K}_i} D(x, p')$. In Klein model with $D = d^R$, we define the function $\sigma_{\mathcal{K}_i}(x) = \sum_{x \in \mathcal{K}_i} \frac{1 - \langle x, p \rangle}{\sqrt{1 - \langle p, p \rangle}}$, and $x \in \text{Vor}_k(\mathcal{K}_i) \iff h_{\mathcal{K}_i}(x) \leq h_{\mathcal{K}_i}(x) \forall j \neq i$. By identifying those hyperplane equations with the generic power diagram hyperplane $h(x) : y = -2\langle x, c \rangle - w + \langle c, c \rangle$ for a ball centered at $c$ and radius $r^2 = w$ ($r$ may be imaginary when $w < 0$), we transform each $k$-subset $\mathcal{K}_i$ in Klein model into a weighted point (or ball) $b(\mathcal{K}_i, w_i): b_i = \sum_{p \in \mathcal{K}_i} \frac{p}{2\sqrt{1 - \langle p, p \rangle}}$. and $w_i = \langle c_i, c_i \rangle - \sum_{p \in \mathcal{K}_i} \frac{1}{\sqrt{1 - \langle p, p \rangle}}$. This method is only practical if when we consider all subsets $\mathcal{K}_i$ that yields non-empty cells, otherwise we have $N = \binom{n}{k}$ too many balls to be tractable!

V. HVDS FROM THE HYPERBOLOID MODEL

Consider the symmetric bilinear form $L = \text{diag}(-1, 1, \ldots, 1)$ in Minkowski space $\mathbb{R}^{1,d}$; $\langle p, q \rangle_L = p^\top L q = -p^{(0)}q^{(0)} + \sum_{i=1}^d p^{(i)}q^{(i)}$. The hyperboloid model is defined on the upper sheet domain $\mathbb{L}^+ = \{\langle x, x \rangle = -1, x_0 > 0\}$ (interpreted as a sphere $(x, x)_L = R^2$ of imaginary radius $R = i$). For $x \in \mathbb{R}^d$, we denote $x_L$ its point obtained by vertically raising $(\cdot, x)$ on $\mathbb{L}^+$: $x_L = (\sqrt{1 + \langle x, x \rangle}, x)$, called Weierstrass coordinates. The hyperbolic distance is expressed by $D^L(p^L, q^L) = \arccosh(-\langle p^L, q^L \rangle)$ and is equivalent to $d^L(p^L, q^L) = -\langle p^L, q^L \rangle_L$. For two points $p^L$ and $q^L$ on $\mathbb{L}^+$, the bisector equation is $\langle x_L, p^L - q^L \rangle = 0$. The bisector is an hyperbola of equation $\langle 1 + \langle p, p \rangle - 1 + \langle q, q \rangle \rangle_{1 + \langle x, x \rangle} + \langle q - p, x \rangle = 0, x \in \mathbb{R}^d$ (for). This hyperbola bisector is contained in a hyperplane $H(p, q) \subset \mathbb{R}^{d+1}$ passing through the origin $O$: $H(p, q) : (\sqrt{1 + \langle p, p \rangle} - \sqrt{1 + \langle q, q \rangle})x_0 + \langle q - p, x \rangle = 0$. The Klein disk model is obtained from $\mathbb{L}^+$ by a central projection $\pi$ from the origin to the hyperplane $H_1 : x_0 = 1; \pi \left[ \begin{array}{c} x_0 \\ x \end{array} \right] = \left[ \begin{array}{c} x \\\ x_0 = \frac{1}{\sqrt{1 + \langle x, x \rangle}} \end{array} \right]$. The disk center touches the apex of $\mathbb{L}^+$. Let $a_{p,q} = \sqrt{1 + \langle p, p \rangle} - \sqrt{1 + \langle q, q \rangle}$. Multiplying (for) by $\frac{1}{\sqrt{1 + \langle x, x \rangle}}$, we have the bisector written as $\langle q - p, x \rangle + a_{p,q} = 0$, an affine bisector in $x'$.

Now consider $\pi_{c,l}$ the generic central projection of $\mathbb{L}^+$ from $C = (c, 0)$ to the hyperplane $H_1 : x_0 = l$ so that $\pi = \pi_{0,1}$. We have $\pi_c \left[ \begin{array}{c} \sqrt{1 + \langle x, x \rangle} \\ x \end{array} \right] = \left[ \begin{array}{c} \frac{l}{\sqrt{1 + \langle x, x \rangle}} - c \\ x \end{array} \right], c \neq 1$. Choosing $c = 0$ and $0 < l \leq 1$ yields the same construction procedure but the clipping of the equivalent power diagram [2] need to be done on a disk of size $l$ since $\|x_{c,l}\| = \|\frac{l}{\sqrt{1 + \langle x, x \rangle}}\| \leq l$, $\forall x \in \mathbb{R}^d$. Note that clipping may destroy many bounded cells of the affine diagram. Thus an open question is to report an optimal output-sensitive construction of the $k$-order HVDS.

REFERENCES
